

# Universal $p$ -ary designs

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## Abstract

We investigate  $p$ -ary  $t$ -designs which are simultaneously designs for all  $t$ , which we call universal  $p$ -ary designs. Null universal designs are well understood due to Gordon James via the representation theory of the symmetric group. We study non-null designs and determine necessary and sufficient conditions on the coefficients for such a design to exist. This allows us to classify all universal designs, up to similarity.

## KEYWORDS

incidence matrix, integral designs,  $p$ -ary designs,  $t$ -designs

## 1 | INTRODUCTION

Designs over the integers have been an object of study for many years, yet the same structures over the ring of integers modulo some prime  $p$  have received little attention. Wilson [7] introduced these  $p$ -ary designs when considering set systems with restricted intersection properties and made a few observation on  $p$ -ary designs for their own sake. These designs, however, turn out to be important in the representation theory of the symmetric group, as null designs characterise certain Specht modules and non-null designs give rise to certain extensions of Specht modules by the trivial module. Despite this, not much work has been done in developing the theory of  $p$ -ary designs. In this paper we shall prove an existence and uniqueness style theorem for universal  $p$ -ary designs, but first we shall briefly recall some definitions and facts from the theory of designs. The reader can find more details in, for example, the textbooks of Dembowski [1] or van Lint and Wilson [6].

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**Definition 1.** Let  $[v] := \{1, 2, \dots, v\}$  be a finite set and  $t \leq s \leq v$  be integers. An integral  $t$ -design on  $v$   $[v]$  of constant block size  $s$  is a function

$$u: [v]_s \rightarrow \mathbb{Z},$$

where  $[v]_s$  is the set of all subsets of  $[v]$  of size  $s$ , such that

$$\hat{u}(Z) := \sum_{Y \supseteq Z} u(Y) = \mu_t \quad \forall Z \in [v]_t$$

We call  $\mu_t$  the *coefficient* of the design and if  $\mu_t = 0$  then we say  $u$  is a *null-design*. We say that  $\hat{u}$  is induced from  $u$  and we shall denote its restriction to sets of size  $j$  by  ${}_j\hat{u}$ .

Similarly a  $p$ -ary  $t$ -design is a function  $u: [v]_s \rightarrow \mathbb{F}$ , a field of positive characteristic  $p$ , such that  $\hat{u}$  is constant on sets of size  $t$ . It is well known that integral  $t$ -design is also an integral  $j$ -design for all integers  $0 \leq j \leq t$ , however this is not true over fields of positive characteristic. If a  $p$ -ary design,  $u$ , of constant block size,  $s$ , is a  $j$  design for all  $j < s$  then we say that  $u$  is a *universal* design for the partition  $(a, b)$ , where  $a = v - s$  and  $b = s$ . The change of perspective here to consider a design as a design for a partition is due to the correspondence between universal designs for  $(a, b)$  and certain extensions of the Specht module  $S^{(a,b)}$ . It is also helpful when stating the existence and uniqueness theorem which is the main result of this paper as this depends on number theoretic properties of this partition.

**Example 2.** Let  $n = 4$ , and define

$$u(Y) = \begin{cases} 1 & \text{if } 4 \notin Y \\ 0 & \text{otherwise} \end{cases}$$

for  $|Y| = 2$ .

It is clear that  $u$  is not an integral 1-design as  $\hat{u}(\{4\}) = 0$ , while  $\hat{u}(\{x\}) = 2$  for  $x \in \{1, 2, 3\}$ . This observation, however, shows that  $u$  is a (null) 2-ary design. As it is also a 2-ary 0-design, with  $\hat{u}(\emptyset) = 1$ , we conclude that  $u$  is a universal 2-ary design for  $(2, 2)$ .

Graver and Jurkat [2] determined when universal integral designs exist.

**Theorem 3** (Graver and Jurkat [2]). *Let  $t \leq s \leq v$  be integers. There exists a universal integral design for  $(v - s, s)$  with coefficients  $\mu_j$  if and only if  $\mu_{j+1} = \frac{s-j}{v-j}\mu_j$  for  $0 \leq j < t$ .*

The goal of this paper is to prove the equivalent result for  $p$ -ary designs and to describe the resulting universal designs, up to similarity. Null universal designs are well understood via a James' kernel intersection theorem, which was proved in the context of the representation theory of the symmetric group, but is re-stated in the language of designs as Theorem 10. The existence, or otherwise, of non-null universal designs depends on the partition. We shall conclude our introduction with a number of definitions required to state the main result, and also some facts on divisibility of binomial coefficients.

**Definition 4.** Let  $u$  and  $u'$  be universal  $p$ -ary designs for  $(a, b)$  with coefficients  $\mu_j$  and  $\gamma_j$  respectively. We say  $u$  and  $u'$  are similar if there is some  $k \in \mathbb{F}$  such that  $\mu_j = k\gamma_j$ .

We shall now state some well-known results on the divisibility of binomial coefficients, as many of the results in the theory of  $p$ -ary designs involve determining whether certain binomial coefficients are  $0 \pmod{p}$  or not.

Let  $a = \sum_{i=0}^{\alpha} a_i p^i$  be the base  $p$  expansion of  $a$ ; that is  $0 \leq a_i \leq p-1$  and  $a_{\alpha} \neq 0$ . The  $p$ -adic valuation  $\text{val}_p(a)$  is the least  $i$  such that  $a_i$  is non-zero, we call  $\alpha$  the  $p$ -adic length of  $a$  and write  $l_p(a) = \alpha$ .

**Definition 5.** Let  $(a, b)$  be a two part partition, that is  $a \geq b > 0$ . We call a partition James if  $\text{val}_p(a+1) > l_p(b)$ , while if  $b = p^{\beta} + \hat{b}$  and  $\hat{b} < p^{\text{val}_p(a+1)} < p^{\beta}$  we call  $(a, b)$  pointed.

So, for example, when  $p = 5$  the partition  $(4, 1)$  is James, the partition  $(29, 26)$  is pointed, and the partition  $(5, 1)$  is neither James nor pointed.

**Lemma 6** (Kummer's Theorem). *Let  $p$  be a prime and  $a, b \in \mathbb{N}$ , then  $\text{val}_p\left(\binom{a+b}{b}\right)$ , the highest power of  $p$  that divides  $\binom{a+b}{b}$ , is the number of carries that occurs when  $a$  and  $b$  are added in their base  $p$  expansions.*

**Lemma 7** (Lucas's Theorem). *Let  $a = \sum_{i=0}^r a_i p^i$  and  $b = \sum_{i=0}^r b_i p^i$ , with  $0 \leq a_i, b_i \leq p-1$ . Then*

$$\binom{a}{b} \equiv \binom{a_0}{b_0} \binom{a_1}{b_1} \cdots \binom{a_r}{b_r} \pmod{p}.$$

*In particular,  $\binom{a}{b} \equiv 0 \pmod{p}$  if and only if some  $a_i < b_i$ .*

**Corollary 8.** *Let  $a, b \in \mathbb{N}$ . The binomial coefficients  $\binom{a+1}{1}, \binom{a+2}{2}, \dots, \binom{a+b}{b}$  are all divisible by  $p$  if and only if  $a \equiv -1 \pmod{p^{l_p(b)}}$ .*

*Remark.* This gives an alternative characterisation of a James partition, in particular  $\lambda = (a, b)$  is James if and only if  $a \equiv -1 \pmod{p^{l_p(b)}}$ , or equivalently  $l_p(b) < \text{val}_p(a+1)$  for all  $i < r$ .

We can now state our main result:

**Theorem 9.** *Let  $a, b \in \mathbb{N}$ , with  $a \geq b$  and let  $u$  be a non-null universal  $p$ -ary design for  $(a, b)$ . If  $(a, b)$  is neither pointed or James, then  $u$  is similar to the constant design. If  $(a, b)$  is James then  $u$  is unique up to similarity, while if  $(a, p^{\beta} + \hat{b})$  is pointed then  $u = u' + c$  where  $u'$  is non-null only as a  $\hat{b}$ -design, while  $c$  is similar to the constant design.*

## 2 | UNIQUENESS OF $p$ -ARY DESIGNS

Universal null  $p$ -ary designs are well understood, due to the work of James on the representation theory of the symmetric group. James' well-known kernel intersection theorem gives a characterisation of the Specht module  $S^{(a,b)}$  as the collection of all null universal  $p$ -ary designs for  $(a, b)$  [3].

**Theorem 10.** Let  $X, Y \subseteq [v]$  with  $|X| = |Y| = b$  and  $X \cap Y = \emptyset$ . Let  $f: X \rightarrow Y$  be a bijection. Define

$$u: [v]_b \rightarrow \mathbb{F}$$

by

$$u(Z) = \begin{cases} (-1)^{|Z \cap Y|} & \text{if } (Z \subseteq X \cup Y) \wedge (\forall x \in X)(x \in Z \Rightarrow f(x) \notin Z) \\ 0 & \text{otherwise.} \end{cases}$$

Then  $u$  is a null  $p$ -ary design for  $(v - b, b)$ . Moreover any null  $p$ -ary design for  $(v - b, b)$  is a linear combination of designs of this form.

Non-null designs also play an important role in the representation theory of the symmetric group, as they determine certain non-split extensions of Specht modules, investigated by the author in [4]. Theorem 3 describes the relationship between the coefficients of integral designs, and a similar analysis determines when a  $p$ -ary  $t$ -design is also a  $j$ -design.

The inclusion matrix,  $A_i^b(v)$ , where  $i \leq b \leq v$ , is the  $\binom{v}{i} \times \binom{v}{b}$  matrix whose rows are indexed by subsets of  $[v] := \{1, 2, \dots, v\}$  of size  $i$  and whose columns are indexed by subsets of  $[v]$  of size  $b$ . The entry corresponding to position  $X, Y$  is 1 if  $X \subseteq Y$  and 0 otherwise. If  $u$  is an integral design for  $(v - b, b)$ , then considering  $u$  as a vector of length  $\binom{v}{b}$ , we see that

$$A_i^b(v)u = \mu_i \mathbf{1}_i,$$

where  $\mathbf{1}_i$  is the vector of length  $\binom{v}{i}$  consisting of 1's. It is clear that

$$A_j^i(v)A_i^b(v) = \binom{b-j}{i-j} A_j^b(v),$$

and thus

$$\binom{v-i}{i-j} \mu_i = \binom{b-j}{i-j} \mu_j,$$

proving the necessity of the conditions in Theorem 3.

**Proposition 11.** Let  $u: [v]_b \rightarrow \mathbb{F}$  be a  $p$ -ary  $t$ -design of block size  $b$  on a set of size  $v$  with coefficient  $\mu_t$ . Let  $j \leq b$  be such that  $\binom{b-j}{t-j} \not\equiv 0 \pmod{p}$ , then  $u$  is also a  $j$ -design, with coefficient

$$\mu_j = \frac{\binom{v-j}{t-j}}{\binom{b-j}{t-j}} \mu_t.$$

*Proof.*

$$\begin{aligned} \binom{b-j}{t-j} A_j^b(v) u &= A_j^t(v) A_t^b(v) u \\ &= A_j^t(v) \mu_t \mathbf{1}_t \\ &= \binom{v-j}{t-j} \mu_t \mathbf{1}_t. \end{aligned}$$

The result follows since  $\binom{b-j}{t-j}$  is invertible in  $\mathbb{F}$ .  $\square$

*Remark.* Wilson [7] showed that there are examples of  $t$ -designs which are not  $j$  designs whenever  $\binom{b-j}{t-j} \equiv 0 \pmod{p}$ , which is very different to the behaviour of integral designs.

In light of this result, to check a design is universal it suffices to check that it is a  $b - p^l$ -design for all  $l \leq l_p(b)$ .

**Proposition 12.** *Let  $\lambda = (a, b)$ . A design for  $\lambda$  is universal if and only if it is a  $(b - p^l)$ -design for all  $l \leq l_p(b)$ .*

*Proof.* Of course a universal design is a  $(b - p^l)$ -design. A  $(b - p^l)$ -design, is also a  $j$  design for all  $j < b - p^l$  with  $\binom{b-j}{b-p^l-j} \neq 0$ ; that is, for any  $j$  such that the sum  $(b - j - p^l) + p^l$  has no carries in  $p$ -ary notation, by Lemma 6. This is precisely those  $j$  for which the coefficient of  $p^l$  in the  $p$ -ary expansion of  $b - j$ , which we shall denote  $(b - j)_l$ , is non-zero. If  $j < b$ , then some  $(b - j)_l \neq 0$ , and as  $u$  is a  $(b - p^l)$ -design  $u$  is also a  $j$ -design by Proposition 11.  $\square$

Wilson has determined when non-null  $p$ -ary  $t$ -designs exist.

**Theorem 13** (Wilson [7]). *Let  $t \leq b \leq v - t$ . Then there is a non-null  $p$ -ary  $t$ -design of block size  $b$  if and only if*

$$\binom{b-i}{t-i} \equiv 0 \pmod{p} \quad \text{implies} \quad \binom{v-i}{t-i} \equiv 0 \pmod{p}$$

for all  $i \leq t$ .

**Corollary 14.** *There are non-null  $p$ -ary  $(b - p^l)$ -designs for  $(a, b)$  if and only if  $a_l \not\equiv -1 \pmod{p}$  or  $b \leq p^{l+1}$ .*

*Proof.* By Theorem 13 a non-null  $(b - p^l)$ -design exists if

$$\binom{b-j}{p^l} \equiv 0 \pmod{p} \quad \text{implies} \quad \binom{a+b-j}{a+p^l} \equiv 0 \pmod{p}.$$

If  $a_l \equiv -1 \pmod{p}$  and  $b > p^{l+1}$  then setting  $j = b - p^l$  we see that non-null designs can not exist. On the other hand if  $b \leq p^{l+1}$  then  $\binom{b-j}{p^l} \not\equiv 0 \pmod{p}$  for all  $j < b - p^l$  so there are non-null  $(b - p^l)$ -designs. Finally, if  $a \not\equiv -1 \pmod{p}$  then  $\binom{b-j}{p^l} \equiv 0 \pmod{p}$  whenever  $(b - j)_l = 0$ . If  $(b - j)_l = 0$  then the sum  $(a + p^l) + (b - j - p^l)$  necessarily has a carry in  $p$ -ary notation, so  $\binom{a+b-j}{a+p^l} \equiv 0 \pmod{p}$  by Lemma 6.  $\square$

Combining this with the relationship between coefficients established in Proposition 11, we obtain more integers  $j$  for which a universal design for  $(a, b)$  is null.

**Proposition 15.** *If a universal design,  $u$ , for  $(a, b)$  is non-null as a  $j$ -design, then  $(b - j)_m + a_m < p$  for all  $m < l_p(b)$ .*

*Proof.* Suppose  $u$  is non-null as a  $j$ -design with coefficient  $\mu_j$ , and let  $m < l_p(b)$  be such that  $(b - j)_m \neq 0$ . As  $u$  is non-null for  $j$ , we must have  $u$  is non-null for  $b - p^m$ , by Proposition 11, as

$$\mu_j = \frac{\binom{a+b-j}{b-p^m-j}}{\binom{b-j}{b-p^m-j}} \mu_{b-p^m}.$$

For  $u$  to be non-null as a  $j$ -design, we must have  $\binom{a+b-j}{b-p^m-j} \neq 0$ . Corollary 14 ensures that  $a_m \not\equiv -1 \pmod{p}$  and thus  $(a + p^m) + (b - j - p^m)$  having no carries is equivalent to  $(a) + (b - j)$  having no carries. Using Lemma 6 we see that if  $u$  is non-null then  $(a) + (b - j)$  has no carries, and therefore  $(b - j)_m + a_m < p$  for all  $m < l_p(b)$ .  $\square$

Our next goal is to determine what the relationship is between the non-zero coefficients of a universal design. Let  $u$  be a universal design for  $(a, b)$ , and let  $X$  be the set of all  $j$  with  $(b - j)_m + a_m < p$  for all  $m < l_p(b)$ . Observe if  $j \notin X$  then  $u$  must be a null  $j$ -design, and so  $X$  contains all  $j$  such that  $u$  is a non-null  $j$ -design. We shall define a partial ordering on  $X$  by setting  $i \geq_X j$  if  $i > j$  and  $\binom{b-j}{i-j} \not\equiv 0 \pmod{p}$ . If  $i \geq_X j$  and  $\mu_i$  and  $\mu_j$  are the coefficients of  $u$

corresponding to  $i$  and  $j$ , respectively, then  $\mu_i = \frac{\binom{a+b-i}{j-i}}{\binom{b-i}{j-i}} \mu_j$ . Following [5] we define a connected

component of a poset to be a connected component of the underlying graph, whose vertices are the elements of the poset, and whose (undirected) edges indicate that two elements are related. If two elements,  $i$  and  $j$ , are in the same connected component of  $X$  there is a relationship between the coefficients  $\mu_i$  and  $\mu_j$  obtained by following a path in the graph between  $i$  and  $j$  and repeated application of Proposition 11. The coefficients of a universal design are determined by the coefficients on each connected component, so the structure of  $X$  gives restrictions on the possible coefficients of a universal design.

**Proposition 16.** *If  $\lambda = (a, b)$  is James, then  $X$  has a single connected component.*

*Proof.* Recall if  $\lambda$  is James then  $b < p^{\text{val}_p(a+1)}$ , and  $a_m \equiv -1 \pmod{p}$  for all  $m < l_p(b)$ . Write  $b = \alpha p^\beta + \hat{b}$ , where  $\beta = l_p(b)$  and  $\hat{b} < p^\beta$ , and observe, by Proposition 15, that  $X = \{\hat{b}, p^\beta + \hat{b}, \dots, (\alpha - 1)p^\beta + \hat{b}\}$ , which, by Lemma 7, is a single connected component.  $\square$

**Proposition 17.** *If  $\lambda = (a, b)$  is not James, and  $b = \alpha p^\beta + \hat{b}$  then  $X$  has a single connected component, unless  $\lambda$  is pointed, in which case  $X$  has two connected components, one of which consists only of the element  $\hat{b}$ .*

*Proof.* Observe that  $i, j \in X$  are comparable if and only if  $(b - i)_m \leq (b - j)_m$  for all  $m \leq l_p(b)$ , or  $(b - j)_m \leq (b - i)_m$  for all  $m \leq l_p(b)$ . Observe also that  $(b - i)_m = 0$  for all  $m < l_p(b)$  for which  $a_m \equiv -1 \pmod{p}$ . The join of  $i, j \in X$ , if it exists, is the element  $i \vee j = x$  such that  $(b - x)_m = \max\{(b - i)_m, (b - j)_m\}$ , the meet,  $y = i \wedge j$ , is the element  $y$  such that  $(b - y)_m = \min\{(b - i)_m, (b - j)_m\}$ . These may fail to be in  $X$  as it may be that  $(b - x) > b$  or  $b - y = 0$ . If, however,  $(b - i)_m$  and  $(b - j)_m$  are both non-zero for some  $m$  then  $i \wedge j \in X$ .

Let  $x$  be such that  $(b - x)_m = p - 1 - a_m$  for  $m < \beta$  and  $(b - x)_\beta = \alpha - 1$ , and observe that  $x \in X$  by Proposition 15. Clearly  $j \in X$  with  $j > \hat{b}$  is comparable to  $x$ . If  $j < \hat{b} \in X$ , or if  $j = \hat{b}$  and  $\alpha \neq 1$  then  $x \wedge j \in X$ .

It only remains to consider the case where  $j = \hat{b}$  and  $\alpha = 1$ , which, if  $\hat{b} > p^{\text{val}_p(a+1)}$  is clearly comparable to  $\hat{b} - p^{\text{val}_p(a+1)}$ , which is in the same component as  $x$ . It follows that if  $\lambda$  is not pointed then there is only one connected component of  $X$ .

On the other hand, when  $\lambda$  is pointed  $\hat{b}$  is not comparable to any other element and thus is in a connected component of its own. This is as no  $j < \hat{b}$  is in  $X$  as no  $j < \hat{b}$  has  $(b - j)_m = 0$  for all  $m < l_p(b)$  where  $a_m \equiv -1 \pmod{p}$ . Similarly no  $j > \hat{b}$  has  $(b - j)_\beta \geq 1$ , so  $j$  and  $\hat{b}$  are incomparable.  $\square$

*Proof of uniqueness in Theorem 9.* If  $u$  is a universal design for  $(a, b)$ , then its coefficients are entirely determined by the connected components of  $X$ , thus an understanding of this poset allows us to determine the possible coefficients of designs. If  $(a, b)$  is not pointed, then non-null universal designs, if they exist, are unique up to similarity, while if  $(a, b)$  is pointed, then any design must be the sum of two designs, uniquely determined by its coefficients on each of the two connected components of  $X$ .  $\square$

### 3 | EXISTENCE OF DESIGNS

In the previous section we have seen a complete characterisation of universal null  $p$ -ary designs and described, up to similarity, the uniqueness of non-null universal designs. We now move to considering the existence of non-null designs for  $(a, b)$ . We first consider  $p$ -ary designs which come from the modulo  $p$  reduction of integral designs. Clearly the constant design,  $c_{(a,b)}$ , is an integral design, with coefficients  $\mu_i = \binom{a+b-i}{b-i}$ , and therefore is null if and only if  $(a, b)$  is James.

**Proposition 18.** *Let  $(a, b)$  be a partition which is neither James nor pointed. Then the constant design is the unique, up to similarity, universal  $p$ -ary design for  $(a, b)$ .*

**Proposition 19.** *Let  $\lambda = (a, b)$ , then there exists an integral design which is not similar to the constant design if and only if  $\lambda$  is James.*

*Proof.* Any integral design must have coefficients satisfying the conditions of Theorem 3,  $\mu_{s+1} = \frac{b-s}{a+b-s} \mu_s$  for  $0 \leq s < t$ . This means that

$$\mu_s = \frac{\binom{a+b-s}{a}}{\binom{a+b}{a}} \mu_0.$$

To ensure that some  $\mu_i \not\equiv 0 \pmod{p}$  we must take  $\mu_s = k \frac{\binom{a+b-s}{a}}{p^d}$  where  $k \in \mathbb{F}$  is non-zero and  $d$  is the least power of  $p$  dividing some  $\binom{a+b-s}{a}$  for  $s \in \{0, 1, \dots, b-1\}$ . That is,  $d = \min_{s < b} \{\text{val}_p \binom{a+b-s}{a}\}$ . Observe that

$$\begin{aligned} {}_j \hat{c}_\lambda &= \binom{a+b-j}{b-j} \cdot c_{(a+b-j,j)} \\ &= k^{-1} p^d \mu_j \cdot c_{(a+b-j,j)}, \end{aligned}$$

and so if  $p^d$  is a unit in  $\mathbb{F}$ , that is if  $d = 0$ , then

$$\psi_{1,j}(k \cdot c_\lambda - u) = 0,$$

and  $u$  is not similar to the constant design. This means  $u$  is similar to the constant design if and only if  $p \mid \binom{a+b-j}{a}$  for all  $j \in \{0, 1, \dots, b-1\}$ , which by Corollary 8 is if and only if  $\lambda$  is James.  $\square$

**Theorem 20.** *The unique, up to similarity, universal  $p$ -ary design for a James partition  $(a, b)$  is the modulo  $p$  reduction of the integral design with coefficients  $\mu_s = \frac{\binom{a+b-s}{a}}{p^d}$  where  $d = \min_{s < b} \{\text{val}_p \binom{a+b-s}{a}\}$ .*

We have seen that if  $(a, b)$  is pointed then the constant design is non-null. We shall now construct another non-null design for  $(a, b)$  which is not similar to the constant design, completing the classification.

**Proposition 21.** *Let  $(a, b)$  be such that  $b = p^\beta$  and  $\text{val}_p(a+1) < \beta$ . Then there exists a universal  $p$ -ary design which is null as a  $t$ -design for all  $t > 0$  and non-null as a 0-design.*



*Proof.* Let  $m = a - b + 1$  and define

$$u(Y) = \begin{cases} 1 & \text{if } Y \cap [m] = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

for  $|Y| = b$ . Then, for  $|Z| = s$

$$\hat{u}(Z) = \begin{cases} \binom{a+b-m-s}{b-s} & \text{if } Z \cap [m] = \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

By our choice of  $m$  the coefficients are  $\binom{a-m+1}{1}, \binom{a-m+2}{2}, \dots, \binom{a-m+b-1}{b-1}$  are all divisible by  $p$ , by Corollary 8. Then  $u$  is a universal design which is non-null only as a 0-design.  $\square$

Let  $u$  be the design constructed above for the partition  $(a, p^\beta)$ . We shall modify  $u$  to construct a design for a pointed partition  $(a, p^\beta + \hat{b})$ , where  $\hat{b} < p^{\text{val}_p(a+1)} < p^\beta$ , which is non-zero only as a  $\hat{b}$ -design.

Let  $u: [v]_{p^\beta} \rightarrow \mathbb{F}$  be the design constructed above for the partition  $(a, p^\beta) \vdash v$ . We shall modify  $u$  to construct a design for a pointed partition  $(a, p^\beta + \hat{b})$ , where  $\hat{b} < p^{\text{val}_p(a+1)} < p^\beta$ , which is non-zero only as a  $\hat{b}$ -design. Let  $Y = \{a + p^\beta + 1, \dots, a + b\}$ , then  $Y$  is a set of size  $\hat{b}$ .

Define  $u_Y: [v]_{p^\beta + \hat{b}} \rightarrow \mathbb{F}$  by

$$u_Y(Z) = \begin{cases} u(Z \setminus Y) & \text{if } Y \subseteq Z \\ 0 & \text{otherwise,} \end{cases}$$

and  $u^Y: [v]_{p^\beta} \rightarrow \mathbb{F}$  by

$$u^Y(Z) = \begin{cases} u(Z) & \text{if } Y \cap Z = \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Given a subset  $X \subset [v]$  we denote by  $\delta_X: [v]_{|X|} \rightarrow \{0, 1\}$  the indicator function; that is

$$\delta_X(Y) = \begin{cases} 1 & \text{if } X = Y \\ 0 & \text{otherwise.} \end{cases}$$

Of course, these functions may not be designs, but we may consider the functions they induce on subsets of  $[v]$  as before. Consider  $\hat{u}_Y: [v]_j \rightarrow \mathbb{F}$ , by grouping terms by the size of their intersection with  $Y$ . First, consider the case where  $\hat{b} < j < b$ :

$$\hat{u}_Y = \binom{\hat{u}}{j-\hat{b}}_Y + \sum_{y \in Y} \binom{\hat{u}}{j-\hat{b}-1}_{Y \setminus \{y\}}^y + \dots + \binom{\hat{u}}{j}_Y.$$

Each of these terms is 0, by our choice of  $u$ , so  ${}_j\hat{u}_Y$  is 0.

Similarly for  $j \leq \hat{b}$

$$\begin{aligned} {}_j\hat{u}_Y &= \sum_{i=0}^j \sum_{|Y' \cap Y|=i} {}_{j-i}(\hat{u})_{Y'} \\ &= \sum_{|Y' \cap Y|=j} ({}_0\hat{u})_{Y'} \\ &= \mu_0 \cdot {}_j\delta_Y, \end{aligned}$$

where  $\mu_0 \neq 0$  is the coefficient of  $u$  as a 0-design.

Observe that if  $Y$  is any subset of  $[a + b]$ , not necessarily  $\{a + p^\beta + 1, \dots, a + b\}$ , then we may define  $u_Y$  as before, by first defining  $u$  on subsets of  $[a + b] \setminus Y$  of size  $p^\beta$ .

Let  $X \subseteq [a + b]$  of size  $b - 1 = p^\beta + \hat{b} - 1$ . Define  $u_X := \sum_{Y \subseteq X} u_Y$ . Then

$$\widehat{u}_X = \sum_{Y \subseteq X} \widehat{u}_Y,$$

which is 0 when restricted to sets of size  $j$  if  $\hat{b} < j < b$ . When  $j \leq \hat{b}$ ,

$$\begin{aligned} {}_j\hat{u}_X &= \sum_{Y \subseteq X} {}_j\hat{u}_Y \\ &= \sum_{Y \subseteq X} \mu_0 \cdot {}_j\delta_Y \\ &= \binom{p^\beta - 1 + \hat{b} - j}{\hat{b} - j} \mu_0 \cdot {}_j\delta_X, \end{aligned}$$

which is 0 if  $j \neq \hat{b}$ . So

$${}_{\hat{b}}\hat{u}_X = \mu_0 \cdot {}_{\hat{b}}\delta_X.$$

If  $\mathcal{U}$  is a non-null  $p$ -ary  $\hat{b}$ -design of block size  $b - 1$  and coefficient  $\alpha$  then setting

$$u_{\mathcal{U}} := \sum_X \mathcal{U}(X) u_X,$$

where the sum is over all sets  $X$  of size  $b - 1$  and  $\mathcal{U}(X)$  is the coefficient of  $X$  in the  $\hat{b}$ -design  $\mathcal{U}$ , we see

$$\begin{aligned} {}_{\hat{b}}\hat{u}_{\mathcal{U}} &= \sum_X \mathcal{U}(X) {}_{\hat{b}}\hat{u}_X \\ &= \sum_X \mathcal{U}(X) \mu_0 \cdot {}_{\hat{b}}\delta_X \\ &= \alpha \mu_0 \cdot {}_{\hat{b}}\delta_X, \end{aligned}$$

and of course

$${}_j\widehat{u}_{\mathcal{U}} = 0$$

for all  $j \neq \hat{b}$ .

**Theorem 22.** *Let  $\lambda = (a, b)$  be such that  $b = p^\beta + \hat{b}$  and  $\hat{b} < p^{\text{val}_p(a+1)} < b$ . Then there is a universal design which is non-null only as a  $\hat{b}$ -design.*

*Proof.* An element of the form  $u_u$  as described above is such a design, it remains to prove such an element exists; that is that there is a non-null  $p$ -ary  $\hat{b}$ -design of block size  $b - 1$ . By Theorem 13, we may construct such a design if (and only if)  $\binom{a+b-1-i}{\hat{b}-i} \equiv 0 \pmod{p}$  whenever  $\binom{b-1-i}{\hat{b}-i} \equiv 0 \pmod{p}$ . Of course  $\binom{b-1-i}{\hat{b}-i} = \binom{p^\beta + \hat{b} - 1 - i}{\hat{b} - i} \equiv 0 \pmod{p}$  for all  $i < \hat{b}$ , so it remains to see that  $\binom{a+b-1-i}{\hat{b}-i} \equiv 0 \pmod{p}$  for all  $i < \hat{b}$ ; that is, that  $\binom{a+p^\beta+j}{j} \equiv 0 \pmod{p}$  for all  $j < \hat{b}$ . This follows from Corollary 8, as  $a + p^\beta \equiv -1 \pmod{p^{l_p(\hat{b})}}$ .  $\square$

**Existence of  $p$ -ary designs.** If  $(a, b)$  is James, then the construction of Graver and Jurkat [2] gives rise to a non-null design. If  $(a, b)$  is not James then the constant design is non-null. If  $(a, b)$  is pointed then Theorem 22 gives a non-null universal design, completing the proof of Theorem 9.  $\square$

## 4 | CONCLUSIONS

We have completely classified universal  $p$ -ary designs, extending the results of Graver and Jurkat to positive characteristic. This result builds on the work of Wilson, who determined precisely when non-null  $p$ -ary  $t$ -designs exist. Non-null universal designs for  $(a, b)$  only exist when the partition  $(a, b)$  is either James or pointed. When  $(a, b)$  is James the design is unique, up to similarity, and is the reduction modulo  $p$  of the integral design constructed by Graver and Jurkat. When the partition is pointed any universal design can be written as the sum of two designs, one similar to the constant design, and the other non-null as a  $t$ -design for only one value of  $t$ .

The main motivation for this study was the relationship between universal designs and certain extensions of Specht modules over fields of positive characteristic, hence the interest in designs over fields of positive characteristic, and in particular over  $\mathbb{Z}/p\mathbb{Z}$ . From this point of view the problem is entirely settled, however, there are a number of design theoretic problems which remain open as any question that can be asked about integral designs has a  $p$ -ary design analogue. For example one may wish to find simple universal  $p$ -ary block designs, those where we restrict the design to taking the values 0 or 1, although the main result of this paper reduces this to the computation task of finding 0-1 vectors amongst the vectors which correspond to universal  $p$ -designs, a which is NP-complete. Similarly one may wish to find designs of smallest support, that is those designs who take non-zero values on the fewest sets. Wilson [7], subject to some additional hypothesis, gives a lower bound for the size of the support of a  $p$ -ary  $t$ -design, which we would like to see extended to universal designs. Again, the results of this paper reduce this problem to a computational problem—that of finding vectors of smallest support within a vector space—however a precise description would be of interest.

The most obvious question one would like to answer is how to extend this study to designs modulo  $n$ , without the restriction that  $n$  is prime. Designs over  $\mathbb{Z}/n\mathbb{Z}$ , or  $n$ -ary designs, are yet

to receive any attention, however many of the techniques used in this paper can be used in their study. The goal should be to develop a theory of  $n$ -ary designs, analogous to the theory of integral designs, with an existence and uniqueness theorem extending the main result of this paper.

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